A SURVEY OF THE SCIENTIFIC WORKS OF N. G. CHETAEV

PMM Vol.24, No.1, 1960, pp. 171-197

Nikolai Gur'evich Chetaev was concerned with essential and difficult problems of analytical dynamics, theory of stability of motion, mathematical physics and the theory of differential equations.

In his scientific activity Chetaev was guided by the opinion that "only those investigations have value which arise from applications... and only those theories are actually useful which result from the consideration of particular cases" (1).*

The investigations of Chetaev are distinguished by the rigor of the formulation of the problem and the irreproachability of its solution. Chetaev, following Liapunov, shared the opinion that "it is not permissible to make use of doubtful reasoning as soon as we are concerned with the solution of a definite problem, whether it be a problem of mechanics or physics, provided only that the problem is accurately stated from the point of view of the analysis. The moment it is stated the problem becomes a problem of pure analysis which is to be treated as such" (2).

Chetaev wrote his papers in the most concise style, here and there even laconically. The reading of his papers, therefore, calls for serious preparation and attention on the part of the reader. The difficulties in reading his papers are the result, also, of the essential difficulties of the problems considered.

In Chetaev's investigations, analytical dynamics, stability of motion and the theory of differential equations are closely interwoven. Therefore, the subdivision of this survey into (A) analytical dynamics, (B) theory of the stability of motion, (C) works on the qualitative methods

---

* Numbers in square brackets refer to the list of papers of N.G.Chetaev; that follows this article; those in parentheses to the references.

(1) A.M. Liapunov, *Pafnutii L'vovich Chebyshev. Soobshch. Kharkov. matem. Obshch (Comm. of the Kharkov Math. Soc.)*. 6, 1895.

(2) A.M. Liapunov, *Obshchaia zadacha of ustoichivosti dvizheniia (General problem of the stability of motion)*. Kharkov, 1892; 2nd ed. Moscow-Leningrad, 1935; Izdat. Akad. Nauk SSSR, 1948; Gos. Tekhn. Teor. Izdat., Moscow-Leningrad, 1950.

in the analysis and (D) works on the applied problems is of a conventional nature only.

A. Analytical Dynamics

The works of Chetaev on analytical dynamics can be subdivided into four sections: Gaussian principle and its modifications, equations of dynamics in terms of group variables, stable trajectories in dynamics and the optical-mechanical analogy.

1. Gaussian principle and its modifications. In 1829 Gauss published a theorem which is known to-day as the principle of Gauss. This theorem was formulated by him as follows: "The motion of a system of particles constrained in any manner and subject to arbitrary influences remains at any instant most consistent with that motion which the particles would have acquired if they became free, i.e. the motion takes place under the least possible constraint if by the measure of the constraint at an instant we understand the sum of the products obtained by multiplying the mass of each particle by the square of its deviation from that position which it would occupy if it were a free particle".

The principle of Gauss attracted the attention of a series of scholars. In particular, Appell and Delassus applied this principle to the investigation of mechanical systems with nonlinear nonholonomic constraints. However, due to their definition of virtual displacements for such systems, the principle of Gauss turned out to be inconsistent with the principle of d'Alembert and Lagrange.

At Kazan' E.A. Bolotov was interested in Gauss' principle. In 1918 he gave the most elegant treatment of this principle for linear nonholonomic systems. Naturally, also, Chetaev's attention was attracted by this principle.

In his paper [4], written while a student, Chetaev applied the Gauss principle to the solution of the most difficult problem concerned with the determination of that branch of the possible branches of equilibrium along which the mass of a rotating liquid in the neighborhood of a point of bifurcation will proceed.*

The principle of d'Alembert and Lagrange results from the axiom which defines smooth constraints, and the contradiction between this principle and that of Gauss arose in analytical mechanics in the process of the growth of new ideas about constraints (passage from linear holonomic to

---

* This paper of Chetaev will be considered later.

nonlinear) which required new ideas about virtual displacements. Chetaev generalized this fundamental concept of analytical mechanics [14], and this generalization permitted him to retain Gauss' principle within the framework of the d'Alembert and Lagrange principle.

Another merit of Chetaev, connected with Gauss' principle, refers to the development of a new approach to the problem of the release of material systems from constraints. As is well-known, Gauss' principle is connected with a particular transformation of material systems, namely, that one which releases material systems from all their constraints. In mechanics many attempts have been made to generalize the Gaussian concept of release, and at the same time also Gauss' principle. Before Chetaev two kinds of releases were considered: complete and partial releases. In the first case, the system is set free from all of its constraints, while in the second the system is released only partially of its constraints. Chetaev proposed to consider as a release of a mechanical system any of its transformations subjected to a definite mathematical algorithm (parametric release of material systems).

In paper [14] a mechanical system with k degrees of freedom is considered, subject to nonholonomic nonlinear constraints depending explicitly on time. The position of the system at the given instant is determined by the orthogonal Cartesian coordinates (x_i, y_i, z_i) or by the generalized independent coordinates $q_1, \ldots, q_k$. The velocities of the particles in the actual motion of the system are

$$x_i' = a_i(t, q_s, q_s'), \qquad y_i' = b_i(t, q_s, q_s'), \qquad z_i' = c_i(t, q_s, q_s')$$
$$(i = 1, \ldots, n;\ s = 1, \ldots, k)$$

where a prime denotes the derivative with respect to time.

Chetaev defines the virtual displacements axiomatically by the expressions

$$\delta x_i = \sum \frac{\partial a_i}{\partial q_s'} \delta q_s, \qquad \delta y_i = \sum \frac{\partial b_i}{\partial q_s'} \delta q_s, \qquad \delta z_i = \sum \frac{\partial c_i}{\partial q_s'} \delta q_s$$

where the δq_s are arbitrary infinitely small quantities.

Now, it is not difficult to show that for such a definition of the virtual displacements Gauss' principle follows from that of d'Alembert, when introduced as a consequence of the axiom defining smooth constraints.

In fact, denote by dx_i', dy_i', dz_i' the changes in the velocities of the particles of the system during an interval dt in the actual motion, and by $\delta x_i'$, $\delta y_i'$, $\delta z_i'$ the changes of the velocities in a conceivable motion, calculated for the same coordinates and velocities at the instant

t as in the actual motion, and finally by $\partial x_i'$, $\partial x_i'$, $\partial z_i'$ the changes of the velocities in the released motion.

Then the principle of d'Alembert and Lagrange gives the equation

$$A_{d\delta} + A_{d\partial} - A_{\partial\delta} = 0$$

where $A_{d\delta} = \sum m_i [(dx_i' - \delta x_i')^2 + (dy_i' - \delta y_i')^2 + (dz_i' - \delta z_i')^2]$ is the measure for the deviation of the motion (d) from that of (δ) during the time dt. The quantities $A_{d\partial}$ and $A_{\partial\delta}$ are defined analogously.

From here immediately follows the well-known theorem of Mach for non-holonomic nonlinear constraints

$$A_{d\partial} < A_{\delta\partial}$$

This theorem contains Gauss' principle as a particular case provided that one takes for the motion (∂) the motion of the system which is completely released from the constraints.

In addition, another theorem is obtained which was first noticed by Chetaev, namely

$$A_{d\delta} < A_{\delta\partial}$$

In this way Chetaev, introducing a new definition for the virtual displacements, this definition being the most general of all the known definitions up to the present date, solved one of the important problems of analytical mechanics.

At the present time Chetaev's definition of the virtual displacements has received general recognition.

The paper by N.E. Kochin "On the release of mechanical systems", in which Chetaev's definition of the virtual displacements is used, has a bearing on Chetaev's work in connection with Gauss' principle.

Next, Chetaev proposed an original modification of the Gaussian principle.

He considers a mechanical system restricted by linear smooth constraints [25] and calculates for this system the work T_μ along the elementary cycle, consisting of the direct conceivable motion (according to Gauss) in the field of forces acting on the system and of the inverse motion in the field of forces, the presence of which would be sufficient for the realization of the actual motion provided that the mechanical system were completely free.

By the application of Gauss' principle it is proved that the work T along an analogous cycle constructed for the actual motion is an extremum of T_μ.

Thus, this theorem is equivalent to Gauss' principle. By means of Carnot's principle in thermodynamics this theorem permits us to widen the nature of the usually-considered mechanical systems. This theorem is also interesting as an immediate modification of an idea of Herman and Euler and developed by Lagrange in his exposition of d'Alembert's principle.

Paper [27] is immediately related to these investigations of Chetaev. It is concerned with the motion of a mechanical system depending upon certain forced variable parameters θ_i, the variations of which are connected with the coordinates x_i, y_i, z_i of the system and are such that they do not admit the hypothesis of very small or adiabatic variation. The system is subjected to ideal constraints, restricting the possible displacements $\delta\theta_i$, δx_i, δy_i, δz_i by means of linear relations.

In the paper, a basic principle of dynamics for such systems is established, the principle of d'Alembert and Lagrange is generalized, and this principle is then modified. It turns out that the work A, calculated along the elementary cycle consisting of the direct actual motion in the field of acting constraints and forces and of the inverse motion in the field of forces sufficient for the realization of the actual motion if the mechanical system were completely free, is a minimum of A_μ, where A_μ is the work calculated along the elementary cycle in a conceivable motion (in the sense of Gauss).

In the case where the actual displacements of a mechanical system are among its possible displacements the theorem of vis viva is obtained. This theorem leads to a series of important consequences, in particular, those relating to the stability of the equilibrium position.

Chetaev's paper "On certain constraints with friction" [67], published in this issue, is of great interest.

Usually, considering systems with friction, the latter, by introduction of friction forces, are reduced to systems with smooth constraints. Such systems are then investigated by the usual methods of mechanics. Chetaev showed that, with sufficiently broad assumptions concerning the friction forces, a general theory of material systems with constraints of the friction type can be developed. Addition of friction forces to the forces acting on the system is not required.

2. *Equations of motion in terms of group variables.* Geometrically, a motion can be interpreted as a transformation of variables.

Transformations can be carried out in various ways. The set of transformations representing the motion possesses particular properties which S. Lie and F. Klein reduced to the concept of a transformation group.

The development of these representations of the motion lead to the establishment of the equations of mechanics in terms of a certain Lie group of infinitesimal transformations. These equations were introduced into mechanics in 1901 by H. Poincaré. Considering a mechanical system with n degrees of freedom, restricted by smooth holonomic stationary constraints and under the action of forces which admit a force function, Poincaré introduces n operators of the transitive group and obtains the differential equations of motion in terms of the new group variables.

In Papers [5, 6], Chetaev considers the same problem as Poincaré, but assumes that the constraints are nonstationary, and determines the position of the system by means of the dependent coordinates $x_1, \ldots, x_r$. Then, infinitesimal operators of a certain intransitive group

$$X_0(f) = \frac{\partial f}{\partial t} + \xi_0^1 \frac{\partial f}{\partial x_1} + \ldots + \xi_0^r \frac{\partial f}{\partial x_r}, \qquad X_i(f) = \sum_1^r \xi_i^j \frac{\partial f}{\partial x_j} \qquad (i = 1, \ldots, n)$$

can be found by means of which the transformation

$$\sum_1^n \eta_i X_i(f)\, dt + X_0(f)\, dt$$

carries the system from the given position into an infinitely near position in the actual displacement while the transformation

$$\sum_1^n \omega_i X_i(f)$$

does the same in a possible displacement.

Further, assuming that the operator $X_0(f)$ commutes with all the $X_i(f)$ and making use of the Hamiltonian principle, the author obtains the equations of motion in the form of Poincaré

$$\frac{d}{dt}\frac{\partial T}{\partial \eta_i} = \sum_{s,k} c_{sik} \frac{\partial T}{\partial \eta_k} \eta_s + X_i(T - U) \qquad (i = 1, \ldots, n)$$

and also in a new canonical form

$$\frac{dy_i}{dt} = \sum c_{sik}\, y_k \frac{\partial H}{\partial y_s} - X_i(H), \qquad \frac{dx_i}{dt} = \sum X_s^i \frac{\partial H}{\partial y_s} + X_0^i(H), \qquad y_i = \frac{\partial T}{\partial \eta_i}$$

$$H = \sum y_i \eta_i - T + U$$

where the c_{sik} are the structural constants of the group. These equations are now called the Chetaev equations in terms of group variables.

Next, is proved the existence of a relative integral invariant of the first order for the system of the equations of motion.

Further, Chetaev establishes a Jacobi-Hamilton type differential equation in partial derivatives

$$X_0(V) + H[t, x_1, \ldots, x_r, X_1(V), \ldots, X_n(V)] = 0$$

which is satisfied by the function of action $V(t, x_1, \ldots, x_r, a_1, \ldots, a_r)$ and shows that if a complete integral of this equation is found, then the solution of the dynamical problem is reduced to the equations

$$\frac{\partial V}{\partial a_i} = b_i, \qquad y_i = X_i(V) \qquad (a_i, b_i = \text{const})$$

Paper [6] ends with a proof of Poisson's theorem which permits us to construct a new integral of the equations of motion provided that two integrals of these equations are known.

In Paper [26], which was published considerably later, a further treatment of this field of analytical dynamics is given. In particular, obtaining for the action function the expression

$$\delta V_0 = \sum \omega_s X_s(V) + \sum \omega_s^\circ X_s^\circ(V) = \sum y_s \omega_s - \sum y_s^\circ \omega_s^\circ$$

where X_s° is the operator X_s applied at the initial instant t_0, Chetaev proves the existence of the linear form

$$\Omega = \sum y_s \omega_s$$

which determines a relative integral invariant of the first order, and the quadratic invariant form

$$\Omega' = \sum [\delta y_s \omega_s] - \sum c_{\alpha\beta s} y_s [\omega_\alpha \omega_\beta]$$

To new problems of analytical dynamics belongs the important problem concerning the construction of a group of possible and actual displacements, when the constraints are given by a differential form.

In this same paper [26] Chetaev introduces the concept of cyclic displacements. The author calls a displacement X_α cyclic if

$$X_\alpha(L) = 0, \qquad (X_\alpha, X_k) = 0 \qquad (\alpha = s+1, \ldots, n, k = 1, \ldots, n)$$

holds, where (X_α, X_k) is a Poisson bracket and $L = T + U$ is the Lagrangian function in terms of the group variables. Under these conditions $r - s$ integrals of the Poincaré-Chetaev equations

$$\frac{\partial L}{\partial \eta_\alpha} = \beta_\alpha$$

can easily be found. For the remaining noncyclic displacements the equations reduce to the form

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \eta_j}\right) = \sum c_{\alpha jk} \eta_\alpha \frac{\partial R}{\partial \eta_\alpha} + \sum c_{\alpha j\gamma} \eta_\alpha \beta_\gamma + X_j(R)$$

where

$$R(t, x_1, \ldots, x_r, \eta_1, \ldots, \eta_s, \beta_{s+1}, \ldots, \beta_k) = L - \sum' \frac{\partial L}{\partial \eta_\alpha} \eta_\alpha$$

If, in addition, $c_{\alpha i\gamma} = 0$, then these equations are the equations of motion of a certain holonomic system, the role of the Lagrangian function being played by the Routh function R.

At the end of Paper [26] Chetaev makes two important observations about the solution of equations of motion in terms of group variables: first, when the group is intransitive, and, second, about the possibility of solving equations of the type of Hamilton-Jacobi in terms of more general functions than the function of action.

This paper [26] of Chetaev determined in many ways the direction of subsequent investigations into the dynamics of mechanical systems in terms of group variables.

In Paper [50] an example is given of the application of the above-mentioned equations to the problem of motion of a similarly changed body. A concrete group of Lie is constructed for such a body, and for the first time the equations of motion are obtained analytically.

In this paper Chetaev acknowledged his debt to his former teacher, the Kazan' geometrician and mechanician, D.N. Zeiliger.

3. *Stable trajectories in dynamics.* In Paper [12] Chetaev, apparently for the first time, briefly pointed out the essential importance of theoretically stable motions and their relation to the actual motions in mechanics.

Let $q_1, \ldots, q_n$ and $p_1, \ldots, p_n$ be, respectively, the generalized coordinates and their conjugate momenta of a holonomic system subject to stationary constraints and forces admitting a force function $U_0(q_1, \ldots, q_n)$.

The coefficients g_{ij} in the quadratic form of the kinetic energy

$$T = \frac{1}{2} \sum g_{ij} p_i p_j$$

will depend only on the coordinates.

The complete integral of the Hamilton-Jacobi equation has the form

$$-ht + V_0(q_1, \ldots, q_n, \alpha_1, \ldots, \alpha_n)$$

The energy constant h depends on non-additive constants $\alpha_1, \ldots, \alpha_n$, and the general solution of the mechanical problem is given by the well-known formulas

$$\beta_i = -t\frac{\partial h}{\partial \alpha_i} + \frac{\partial V_0}{\partial \alpha_i}, \qquad p_i = \frac{\partial V_0}{\partial q_i} \qquad (i = 1, \ldots, n) \tag{3.1}$$

where the β_i are new constants of integration.

If the Hamilton function $H(q_i, \ldots, q_n, p_1, \ldots, p_n)$ has the meaning of the total energy $T - U_0$, then the Hamilton canonical differential equations of motion

$$\frac{dq_s}{dt} = \frac{\partial H}{\partial p_s}, \qquad \frac{dp_s}{dt} = -\frac{\partial H}{\partial q_s} \tag{3.2}$$

have the variational equations of Poincaré in the form

$$\begin{aligned} \frac{d\xi_i}{dt} &= \sum_j \frac{\partial^2 H}{\partial p_i \partial q_j}\xi_j + \sum_j \frac{\partial^2 H}{\partial p_i \partial p_j}\eta_j \\ \frac{d\eta_i}{dt} &= -\sum_j \frac{\partial^2 H}{\partial q_i \partial q_j}\xi_j - \sum_j \frac{\partial^2 H}{\partial q_i \partial p_j}\eta_j \qquad (i = 1, \ldots, n) \end{aligned} \tag{3.3}$$

Fixing the constants $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$, any motion can be assumed for the unperturbed motion. The problem of stability of this motion then can be formulated with respect to the coordinates $q_1, \ldots, q_n$ under the condition that the constants $\alpha_1, \ldots, \alpha_n$ do not undergo variations. By virtue of this condition it then follows, from (3.1) that, to within small quantities of the second order,

$$\eta_i = \sum_j \frac{\partial^2 V_0}{\partial q_i \partial q_j}\xi_i$$

This permits us to write the first group of Equations (3.3) by taking into account the relation

$$H = \frac{1}{2}\sum g_{ij}p_i p_j - U_0$$

in the form

$$\frac{d\xi_i}{dt} = \sum_{js} \xi_s \frac{\partial}{\partial q_s}\left(g_{ij}\frac{\partial V_0}{\partial q_j}\right) \tag{3.4}$$

Noticing further that by virtue of the structure of Equations (3.2) the stability of the considered motion in the first approximation is possible only for the zero values of the Liapunov characteristic numbers of the solutions of these equations, Chetaev concludes that a necessary condition for the stability is

$$\chi\left\{\exp \int L\,dt\right\} = 0 \qquad \left(L = \sum_{ij} \frac{\partial}{\partial q_i}\left[g_{ij}\frac{\partial V_0}{\partial q_j}\right]\right) \tag{3.5}$$

where χ is the characteristic number of the function in parantheses. The system is assumed to be regular (2) "as is natural to assume if we are dealing with Nature..." [12].

The set of unperturbed motions with fixed constants $a_1, \ldots, a_n$, Chetaev calls a packet. Further, pointing out the difficulties of judging the stability by the first approximation in problems of mechanics, he introduces potential perturbation forces and states the requirements for stability as follows:

"In Nature it is natural to assume that the perturbation forces admit a force function W depending on the variables q_i. The perturbation forces tend to increase the value of the function W; their influence on the packet at an arbitrary point q_s of the phase space is proportional to the density of the trajectories at this point

$$A^2 = \psi\psi^*$$

"From this it follows that the perturbing forces disturb that packet relatively less, for which

$$\int W\,\psi\psi^*\,d\tau \text{ — maximum} \tag{3.6}$$

where $d\tau$ denotes a volume element of the phase space. This means that considering the set of all motions, the perturbing forces assign absolute stability to that packet which satisfies the condition (3.6). The trajectories of this packet will be called permissible orbits. To make comparisons possible assume for the measurement of the density the natural assumption

$$\int \psi\psi^*\,d\tau = 1$$

"In order to determine the differential equation of the variational problem (3.6), consider that motion of the material system which would have taken place under the same initial data but, in addition, under the action of the perturbation forces. Here the energy integral always exists

$$T = W + U_0 + h$$

"This allows us to write the integral (3.6) in a different form

$$\int (T - U_0 - h)\,\psi\psi^*\,d\tau$$

where in T, instead of the variables p_i, the derivatives $\partial v/\partial q_i$ must be substituted, corresponding to the unperturbed motion. If the expression of the density function

$$\psi = Ae^{iV}$$

is taken into account, then we can conclude that

$$2T\psi\psi^* = \sum g_{ik}\frac{\partial\psi}{\partial q_i}\frac{\partial\psi^*}{\partial q_k} - \sum g_{ik}\frac{\partial A}{\partial q_i}\frac{\partial A}{\partial q_k}$$

"Consequently, the integral (3.6) can be written as follows:

$$-\frac{1}{2}\int\left[-\sum g_{ik}\frac{\partial\psi}{\partial q_i}\frac{\partial\psi^*}{\partial q_k}+\sum g_{ik}\frac{\partial A}{\partial q_i}\frac{\partial A}{\partial q_k}+2(U_0+h)\psi\psi^*\right]d\tau$$

"From here, after obvious transformations, we obtain the following relation

$$\int\delta\psi^*\left[\Delta\psi+2(U_0+h)\psi-\frac{\Delta A}{A}\psi\right]d\tau \doteq 0$$

for the determination of the differential equation of the variational problem (3.6). Whence the basic equation for the permissible orbits is

$$\Delta\psi+2(U_0+h)\psi-\frac{\Delta A}{A}\psi \doteq 0 \qquad (3.7)$$

"If $\Delta A = 0$, then our basic equation (3.7) assumes the form of the differential equation on which Schrodinger in his "Abhandlungen zur Wellenmechanik" has based his wave mechanics."

Remarking casually that the regularity of the solution of Equation (3.7) leads to the eigenvalues of the constants $a_1, \ldots, a_n$ (i.e. also of h), and consequently, to a discrete disposition of stable trajectories, Chetaev concludes this paper [12] in the following way:

"We think about a material system moving under the action of certain forces in a weak field of perturbations. This latter destroys any motion, provided only that it is not stable and permissible. In this way stable and permissible motions are preserved. But it can never be assumed that in Nature the motion takes place along a stable trajectory. There always exist small deviations due to which the actual motions of a material system take place in a sufficiently small domain enveloping a stable trajectory

$$|\xi_i| < \varepsilon$$

"Adjacent trajectories, differing as little as desired from a stable trajectory, must 'oscillate' around the latter ($\kappa_s = 0$)*; this phenomenon gives us an idea of a 'wave'".

Passing to Paper [19] by Chetaev "On stable trajectories of dynamics", let us quote in full, first of all, that part at the beginning of the paper which is basic to the author's conclusions connected with his principal approach to the problems of the stability of motion:

"How are the laws of nature found?

* Here κ_s denotes the characteristic number of the solution of system (3.4).

"To explain any mechanical phenomenon of Nature we first make definite hypotheses about the essential moving forces. This permits us to write down certain differential equations of motion in terms of the variables x_s of the material system under consideration. In the case, where the solutions of these differential equations give for the functions studied ϕ_k values which are near the experimental data (to within the limits of the errors of the experiment), the hypothesis is assumed as a law of Nature, at least until experiments reveal new facts which contradict it. When such facts are discovered, new hypotheses are made without any restriction by the customary fundamental concepts which hold at that time. This is only done provided that in the framework of the latter it is impossible to obtain good agreement with the experiment.

"When can the deviations of the theory from the experiment be insignificant?

"Every time that we make an attempt to explain these or any other phenomena of nature, we must not forget that in reality no phenomena present themselves in a pure form. No matter how precisely the forces acting on the system are determined, there will always exist weak perturbations which have not been taken into account. These latter, no matter how small they may be, influence the motion of the material system and give to the functions, the values of which are determined experimentally, not the theoretical values ϕ_k but certain other values F_k.

"Assume that for the perturbation forces of a certain type and for small perturbations of the initial data, not exceeding numerically a certain small quantity ϵ, the inequality

$$\sum (F_k - \Phi_k)^2 < L$$

holds for all t exceeding the initial instant t_0. Further assume that for an arbitrary number L there always exists a small number ϵ different from zero. Then the unperturbed (theoretical) motion of the mechanical system subject to the given perturbation forces is said to be stable with respect to the functions ϕ_k and unstable in the opposite case.

"In reality, according to this definition of stable and unstable motions, the general character will be preserved, at least with respect to the functions ϕ_k, only by those theoretically unperturbed motions which are stable with respect to ϕ_k. The last circumstance does not mean that all motions, determined by the accepted laws, will turn out to be stable for any small perturbation forces and for arbitrarily small perturbations of the initial data. It means that these laws, due to the basic requirement of small deviations from the experimental data, cannot rely on anything other than the motions which are stable in one or the other measure with respect to the observable functions ϕ_k.

"This proposition, which is a simple consequence of the definition of stable unperturbed motions and of the requirement of small deviations between theory and experiment, and which refers more to the structure of our scientific knowledge, we shall call the postulate of stability and will accept it without reservation. It will not matter whether later on this postulate is confirmed or refuted; for the present it is interesting to see that consequences can be deduced from it."

Further, repeating the statement of the problem of stability of a mechanical system, which was mentioned in the review of Paper [12], and writing out the same equations including the system (3.4), Chetaev gives a rigorous proof to the effect that in the case of stability in the first approximation of the unperturbed motions under consideration, the Poincaré variational equations have only zero characteristic numbers. The proof makes use of the invariant of equations (3.3) given by Poincaré

$$\sum_s (\xi_s \eta_s - \eta_s \xi_s')$$

where $\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n$ and $\xi_1', \ldots, \xi_n', \eta_1', \ldots, \eta_n'$ are two arbitrary solutions of these equations, as well as of the basic lemmas of Liapunov (2) on characteristic numbers.

The assumption that the system (3.4) is regular leads to the condition (3.5). In addition, assume that this system satisfies the requirements of reducibility and that the corresponding linear transformation

$$x_i = \sum_j \gamma_{ij} \xi_j$$

possesses the determinant which is constant and different from zero. Then, due to the invariance of the characteristic numbers of the solutions of the system (3.4) under such a transformation and the well-known theorem of Ostrogradskii-Liouville, we shall obtain from (3.5) the necessary condition of stability in the form

$$L = \sum_{ij} \frac{\partial}{\partial q_i}\left(g_{ij}\frac{\partial V_0}{\partial q_j}\right) = 0 \tag{3.8}$$

This condition expresses the fact that the sum of the characteristic numbers of the system (3.4) is equal to zero.

In the actual motion, let the material system be under the action of forces with the force function U_0, theoretically taken into account above, and subject to unknown perturbation forces which, however, are assumed to be potential forces admitting a force function W. The actual field of forces is then determined by the function $U_0 + W$.

If the statement of the problem of stability for the actual unperturbed motions, under the perturbations of the initial data only, is preserved in the same form as above in the theoretical field of force with the force function U_0, then the necessary requirement for the stability in the first approximation, as, for example, in the form (3.8), will not be effective in the general case, since the function V, playing in the actual motion the role of V_0, is not known (as also is W). However, conditions of stability can be found which do not depend explicitly on the unknown function of the perturbation forces W, but will contain only the constant of integration h which has the independent physical meaning of the total energy.

Let us begin from the requirement of stability in the form (3.8), assuming that the conditions for its existence (reducibility and so on) are satisfied for the actual motions.

Introduce instead of V a new function

$$\psi = Ae^{ikV}$$

where k is a constant, A a real function to be determined and $i = \sqrt{-1}$. After simple calculations, using equations of the type (3.1) and the energy integral for the actual motions, the condition (3.8) assumes the form

$$\frac{1}{\psi}\sum\frac{\partial}{\partial q_i}\left(g_{ij}\frac{\partial\psi}{\partial q_j}\right)-\frac{1}{A}\sum\frac{\partial}{\partial q_i}\left(g_{ij}\frac{\partial A}{\partial q_j}\right)- \\ -\frac{2}{A}\sum g_{ij}\frac{\partial A}{\partial q_j}\left(\frac{1}{\psi}\frac{\partial\psi}{\partial q_i}-\frac{1}{A}\frac{\partial A}{\partial q_i}\right)+2k^2(U_0+W+h)=0 \qquad (3.9)$$

This equation will not contain W if A is determined by the equation

$$-\frac{1}{A}\sum_{ij}\frac{\partial}{\partial q_i}\left(g_{ij}\frac{\partial A}{\partial q_j}\right)-\frac{2ki}{A}\sum_{ij}g_{ij}\frac{\partial A}{\partial q_j}p_i+2k^2W=0$$

which after its separation into real and imaginary parts decomposes into two equations

$$W=\frac{1}{2k^2A}\sum_{ij}\frac{\partial}{\partial q_i}\left(g_{ij}\frac{\partial A}{\partial q_j}\right), \qquad \sum_{ij}g_{ij}\frac{\partial A}{\partial q_j}p_i=0 \qquad (3.10)$$

Equalities (3.10) determine the structure of the perturbation forces for which one of the stability conditions does not depend on these forces explicitly, but depends only by means of the energy constant h. If conditions (3.10) are satisfied, condition (3.9) assumes the form

$$\sum_{ij}\frac{\partial}{\partial q_i}\left(g_{ij}\frac{\partial\psi}{\partial q_j}\right)+2k^2(U_0+h)\psi=0 \qquad (3.11)$$

Single-valued and continuous solutions of Equations (3.11) for the function ψ are admissible only for the eigenvalues of h. Consequently, also, the stability of the actual motions will take place only for these values of the energy constant h. For the function U_0 of the theoretical forces these values of h can be determined in principle from Equations (3.11). Let us quote in full this important part of the paper [19]:

"Because of this kind of effectiveness the method of solving our problem changes sharply to the opposite. Let us imagine our previous material system and assume that it is subject to perturbation forces with the force function W, determined by formulas (3.10). Knowing in advance the force function of the essential or theoretical forces U_0, we can find the eigenvalues of the constant h in the differential equation (3.11). Let ψ be a certain eigenfunction of this equation corresponding to the constant h. If now, in Equation (3.11), the function ψ is replaced by a new function S, determined by the formula

$$\psi = Ae^{ikS}$$

then, separating the real and imaginary parts, this formula, in accordance with the assumption about the structure of the perturbation forces, decomposes into two equations. The first one

$$\frac{1}{2}\sum_{ij} g_{ij}\frac{\partial S}{\partial q_i}\frac{\partial S}{\partial q_j} = U_0 + W + h$$

shows that S will be a particular solution of the Hamilton-Jacobi equation corresponding to the actual motions of the considered material system. The second, existing if this particular solution S appears in the complete Jacobi integral V for the actual motion,

$$\sum_{ij}\frac{\partial}{\partial q_i}\left(g_{ij}\frac{\partial S}{\partial q_j}\right) = 0$$

shows that the necessary condition of stability $\Sigma\,\kappa_s = 0$ is always satisfied.

"If the actual motions are not additionally constrained, then the possibility of obtaining stable motions outside of the solutions just found is not excluded. It is easy to observe that all stable actual motions which are not obtained by this method will have one general property, namely, for them the necessary condition of stability $\Sigma\,\kappa_s = 0$ is not equivalent to condition (3.8).

"If, however, under these circumstances the actual motions are such that the variational equations (3.4) are reducible by means of a substitution with a constant determinant, then according to the previous analysis the possibly stable motions of such a system will be contained in the set of the obtained solutions. Of course, the latter may contain spurious or superfluous solutions which can be discarded if one considers the whole set of necessary conditions for the stability in the first approximation, and does not restrict himself by the single condition

$\sum \kappa_s = 0.$"

In Paper [19] is given the simplest example of a free material particle moving in the field of potential forces with the function U_0. The conditions (3.10) for the structure of the perturbation forces assume the form

$$W = \frac{1}{2k^2 mA} \triangle A, \qquad \sum \frac{\partial A}{\partial q_j} p_j = 0$$

and the condition (3.11) the form

$$\triangle\psi + 2k^2 m (U_0 + h)\psi = 0 \tag{3.12}$$

i.e. it coincides with the well-known Schrodinger equation of quantum mechanics. The latter, in the given case, represents the relation, restricting the choice of the constants in the complete Jacobi integral.

In the case of more complicated necessary conditions of stability $\kappa_j = 0$ (and not only $\sum \kappa_j = 0$) and preserving the reducibility of Equation (3.4), the problem of selecting stable actual motions reduces anew to theorems on the existence of regular solutions $\psi_j^{(sr)}$ of certain systems of differential equations in partial derivatives having, however, in this case, a considerably more complicated form. In the general case, when to the characteristic roots μ_s of the reduced system of differential equations obtained from system (3.4), there correspond arbitrary elementary divisors, the form of these equations is

$$\frac{\partial \psi_j^{(sr)}}{\partial t} + \sum \frac{\partial \psi_j^{(sr)}}{\partial q_i} g_{ik} \frac{\partial V}{\partial q_k} + \sum \psi_i^{(sr)} \frac{\partial}{\partial q_j}\left(g_{jk}\frac{\partial V}{\partial q_k}\right) = \mu_s \psi_j^{(sr)} - \psi_j^{(s,\, r-1)}$$
$$(r = 1, \ldots, n_s;\ s = 1, \ldots, k;\ j = 1, \ldots, n) \tag{3.13}$$

where k is the number of groups of solutions, n_s the number of solutions in a group, corresponding to the root $\mu_s (n_1 + n_2 + \ldots + n_k = n)$, $\psi_j^{(s0)} = 0$, and V is the function contained in the complete Hamilton-Jacobi integral and satisfying the equation

$$\sum g_{ij} \frac{\partial V}{\partial q_i} \frac{\partial V}{\partial q_j} = 2(U_0 + W + h)$$

The function W presupposes a certain structural definiteness.

In the case of simple elementary divisors Equations (3.13) simplify considerably.

The paper concludes with an example and a discussion of the types of the solutions of equations of the form (3.11) for the motion of free particles.

Paper [20] "Stability and the classical laws", published by Chetaev in 1936, is also similar in aim to the paper "On stable trajectories of dynamics". By means of concrete laws of physics Chetaev illustrates the validity of the stability postulate, i.e. the necessity of recognizing the stability of one or the other type (in the sense of selecting functions which take part in experimental measurements, and the form of the perturbation forces) by virtue of the requirement of small deviations of the theory from the experiment.

1. Consider the equilibrium of an isotropic continuous medium, assuming that the inner forces, developed as a result of its deformation, are conservative, and that that part of them which cannot be taken into account (perturbation forces) is not of lower order than two with respect to the small deformations. Of what kind must these inner forces be if one starts with the postulate of stability?

By virtue of Lagrange's theorem on the stability of the equilibrium and its converse by Liapunov and Chetaev [23], there is at every point of the medium a force function of the form

$$U = -k^2 (x_1^2 + x_2^2 + x_3^2) + W \tag{3.14}$$

where x_1, x_2, x_3 are the deviations of this point from the equilibrium position, and W is a function which with respect to these deviations is of the order greater than two.

Thus, the force of elasticity will be defined in accordance with Hooke's Law which has a good experimental foundation.

"It is interesting to note that Hooke's Law does not possess dynamical stability for arbitrarily small perturbation forces (the order of smallness being larger than one). Therefore, from the point of view of the stability postulate, it becomes clear why serious objections have been raised to Hooke's Law on the grounds of its insufficiency in certain dynamical problems" [20].

2. The behavior of the entropy S of a set of bodies, changing in a certain physical and chemical process according to the second law of thermodynamics, is characterized by its nondecrease. If S_0 is its maximum, then

$$\frac{dV}{dt} \geqslant 0$$

where $V = S - S_0$ plays in this law the role of the Liapunov function in his basic theorem on the stability of motion, although here it is impossible to give for the process a clear mechanical analogy.

3. Consider the last problem, which refers to the law of gravitation of Newton and is connected, due to its origin, with the laws of Kepler, which in turn are based on the astronomical observations of Tycho de Brahe.

From the point of view of the Chetaev principle all three laws of Kepler must contain directly such elements of the planetary motion which by necessity are stable in the theoretical law of gravitation of Newton.

Let us quote in full the concluding words, referring to this idea.

"Let us verify! The elements of the first law of Kepler (plane and the law of areas) are obviously stable not only in the law of Newton but also for arbitrary central forces. In the problem of two bodies, if the particle under consideration describes, according to the Newtonian law, an elliptic trajectory, the motion will be stable with respect to the quantity

$$r - \frac{p}{1 + e\cos\varphi}$$

where p and e denote, respectively, the parameter and the eccentricity of the ellipse, described by the particle in the unperturbed motion. Here r and ϕ are the radius vector of the particle in the perturbed motion and the angle between this radius vector and its smallest value in the unperburbed motion. This proposition of Liapunov ("General problem", p. 13) shows that in the second law Kepler also used stable elements. The fact that in the third law Kepler talks about stable elements was established by Laplace, Lagrange and Poisson in the well-known theorem on the stability of the major semi-axes of elliptic orbits".

4. The optical-mechanical analogy. A large and very important part of Chetaev's work is connected with the investigation of the general properties of the perturbed motions of mechanical systems in the neighborhood of a stable unperturbed motion.

Paper [34] occupies an important place in this field: it deals with the properties of the perturbed motions described by the variational equations (3.3).

Here, a fundamental theorem is established to the effect that in the case of a stable unperturbed motion the variational equations (3.3) not only have all their characteristic numbers equal to zero but are also reducible in the sense of Liapunov (2) and possess a definite quadratic integral.

These results permitted Chetaev to pave the way for the development of the optical-mechanical analogy which he completed in his papers [55, 59, 61, 65].

The importance of the optical-mechanical analogy in the development of classical mechanics is well-known. The analogy between the principles of Fermat and Maupertuis, in particular the analogy between the wave theory of light by Huygens and the motion of a conservative mechanical system, played an important role in analytical dynamics. According to Chetaev "... the roots of the beautiful results obtained in analytical dynamics after Lagrange are to be found in the analogy between mechanics and optics. For contemporary problems this analogy does not, in my opinion, play any lesser part".*

Chetaev has underlined that the analogy to the oscillatory process in physics must be sought in the small perturbed motions about a stable motion of a holonomic conservative dynamical system. Thus, in Paper [61], it is said: "Hamilton discovered the analogy between the wave optics of Huygens and the motion of a mechanical system, restricted by holonomic constraints and subject to the action of forces, admitting a force function. This important discovery determined for a century the progress of analytical dynamics".

These remarks illuminate Chetaev's interests and the general trend of his investigations along the lines of the optical-mechanical analogy. Let us examine briefly the paper "On the continuation of the optical-mechanical analogy" [61].

Let us return to Equation (3.8)

$$\sum \frac{\partial}{\partial q_i}\left(g_{ij}\frac{\partial V}{\partial q_j}\right) = 0$$

This equation is of the elliptic type since the g_{ij} are the coefficients of a positive quadratic form which determines the vis viva T.

By virtue of the Hamilton-Jacobi equation the function V satisfies the equation

$$\sum g_{ij}\frac{\partial V}{\partial q_i}\frac{\partial V}{\partial q_j} = 2(U+h) \tag{4.1}$$

Consider now the twice differentiable function

$$\Phi(-ht+V)$$

Under the assumption that the above introduced necessary conditions of stability are satisfied, the function Φ will satisfy the equation

$$\frac{2(U+h)}{h^2}\frac{\partial^2\Phi}{\partial t^2} = \sum \frac{\partial}{\partial q_i}\left(g_{ij}\frac{\partial \Phi}{\partial q_j}\right) \tag{4.2}$$

---

* This quotation is taken from N.G. Chetaev's paper "Dialectical principle and exact natural science" published in 1930 in the *Vestnik Kazanskogo Fiziko-matem. Student. krushka* (lithographed).

"This wave equation establishes the analogy between the mathematical theory of light by Cauchy and the stable motions of holonomic conservative systems".

The optical-mechanical analogy is fully investigated by Chetaev in the light of the theory of Lie groups. The basic idea is the coincidence of the transformation group of one phenomenon (oscillatory process of the propagation of light) with the transformation group of the other (perturbed motions near a stable motion of a mechanical system).

B. Theory of Stability of Motion

For convenience of exposition, Chetaev's work on stability can be subdivided into the following sections: the problem of existence of stable equilibrium figures of rotating liquids; the general theorem on instability and the converse of the theorem of Lagrange; investigation of stability in the first approximation for a non-stationary motion; elaboration of effective methods for the construction of the Liapunov functions.

5. *Stable equilibrium figures of rotating liquids*. The works of Liapunov on the equilibrium figures of a rotating liquid and their stability contain a rigorous proof of the existence of new equilibrium figures uniformly rotating about a certain axis, the liquid being assumed to gravitate according to the Newtonian law, and also the statement together with a solution of the stability problem of these figures. Liapunov proved the instability of pear-shaped figures. This refuted Darwin's cosmogonical hypothesis on the development of an estinguished star through pear-shaped figures of equilibrium. After the works of Liapunov the problem of the development of an ideal extinguished star remained open.

In Papers [3,4], Chetaev set himself the task of investigating a continuous sequence of stable equilibrium figures of a homogeneous rotating liquid mass, subject to the action of the Newtonian gravitation forces, forces of radial compression toward the center of gravity with constant velocity η, and constant pressure on the surface.

First, he showed that the problem of finding the equilibrium figures of such a rotating liquid mass reduces to the solution of the functional equation

$$f\rho \int_{\tau} \frac{d\tau'}{\Delta} + \frac{\omega^2}{2}(x^2+y^2) - \frac{\eta^2}{2}(x^2+y^2+z^2) = \text{const на } (S_t) \tag{5.1}$$

where τ is the volume of the liquid, S_t the free surface, Δ the distance between any two points (x, y, z) and (x', y', z') of the liquid, f the constant of gravitation, ω the angular velocity of rotation of the liquid and ρ the density.

Further, Chetaev proved that the ellipsoids of revolution and the three-axial ellipsoids satisfy Equation (5.1) under certain restrictions. Here, the axis of rotation must be shortest of the axes of the ellipsoid.

Applying the Gaussian principle of least constraint, Chetaev proved that under the action of the forces of attraction and radial compression the angular velocity of rotation of the liquid mass in the actual change of the equilibrium figure is forced to vary in the least way among all conceivable motions. From this it unquestionably follows that in the course of time the mass of the liquid changes its boundary figure of equilibrium in such a way that of all the positions consistent with the constraints the absolute value of the actual variation of the moment of inertia of the liquid with respect to its axis of rotation is the least.

Consequently, the actual figure of equilibrium in the region of a certain bifurcation point will be that for which the inertia moment of the mass

$$\int_{\tau} (x^2 + y^2)\, dm$$

assumes a maximum.

In order to single out a stable sequence of equilibrium figures Chetaev makes use of the Lagrange theorem on stability when a force function exists and adapts the proof suitably for the case under consideration.

Applying this theorem to the linear approximation, Chetaev establishes the distribution of stability and instability in the sequence of ellipsoidal figures of equilibrium of a rotating homogeneous liquid.

Further, stable figures are sought which are derivatives of stable ellipsoids of revolution.

As was mentioned above in connection with the problem of the equilibrium figures of a rotating liquid, at the beginning of the present century serious differences of opinion arose between Liapunov, Poincaré and Darwin as to the question of stability of the pear-shaped figures. The dispute was solved in favor of Liapunov.

However, as Chetaev pointed out in Paper [9], Liapunov's ingenious method overlooks one delicate point still to be considered. As is well-known, Liapunov proposed to consider a certain linear sequence of figures (f), differing as little as desired from the critical ellipsoid E_0 of Jacobi. Separate figures f of this sequence are completely determined by the values of a certain parameter α, and those of them for which certain functions $L(\alpha)$ vanish turn out to be equilibrium figures being the derivatives of the ellipsoid E_0. Since the various f-figures do not represent all geometrically possible figures which are near E_0, then the question whether all the equilibrium figures which are derivatives of E_0 are among the f-figures of Liapunov is of the highest importance.

Paper [9] by Chetaev, which consists of five chapters is devoted to the solution of this difficult problem.

In the first chapter are derived the basic nonlinear integral equations for a variable which determines a figure of equilibrium which is near to the ellipsoid and has the same angular velocity of rotation as the ellipsoid. Chetaev uses here some results of Liapunov, but obtains the basic equation in a form which is slightly different from that of Liapunov and much simpler.

The second chapter is devoted to the investigation of the problem of the distribution of the critical equilibrium figures in the sequence of the Jacobi ellipsoids.

In the third chapter, the author proves that not every figure of equilibrium, being a derivative of the ellipsoidal figures, is among the f-figures of Liapunov. Because of the difficulties in applying the general method for the investigation of the ramification of the solutions of nonlinear integral equations to the problem on the spreading of the equilibrium figures which are the derivatives of the ellipsoids, the author proposed a generalization of the Liapunov method, by means of which he then proved the above-mentioned assertion.

In connection with this there arose the problem of the determination of the sequence of stable equilibrium figures. Chetaev is concerned with this problem in the fourth chapter of his memoir. First, he outlines Liapunov's theorem on the stability of the equilibrium figures, according to which, if for a certain form C of the liquid the function

$$2\Pi = \frac{1}{\rho}\iint\frac{d\tau d\tau'}{r} - \omega^2 \int(x_2 + y^2)d\tau \tag{5.2}$$

assumes a maximum for the given value L of the moment of momentum, then this figure C is stable.

For the case $L \neq 0$ Liapunov showed that it makes no sense to speak about the absolute maximum of the function Π, if the liquid mass is subject to no additional restrictions. Chetaev introduces such an additional condition and proves that, if there exists a lower bound, which is not infinitely small, for the masses of the separate bodies into which a certain homogeneous mass of an incompressible liquid can be decomposed under the influence of the Newtonian forces of attraction and the centrifugal forces, then for this mass there exists at least one body for which Π assumes its largest value, and, consequently, at least one stable equilibrium figure.

The fifth chapter is devoted to the consideration of stability of the equilibrium figures derived from ellipsoids. Chetaev proves here two important general theorems on the number of real branches of the equilibrium curve of a mechanical system, passing through a bifurcation point, and on the change of stability. Particular cases of these theorems were noticed in 1885 by Poincaré.

In order to clarify the problem of the distribution of the stable branches of the equilibrium figures near a critical ellipsoid, the author applies these theorems and proves the existence of a stable sequence of equilibrium figures, being the derivative of the critical MacLaurin ellipsoid and expanding in the direction of large values for the angular velocity of rotation. The chapter concludes with a statement of the principal problem of stability of the Jacobi ellipsoids in the sense of Liapunov.

6. General theorem on instability and the converse of Lagrange's theorem. The other problem which attracted the attention of Chetaev at the beginning of his scientific activity was the celebrated problem of the converse of the Lagrange theorem on the stability of the equilibrium when the force function has a maximum [11, 16, 17, 23, 48].

As is well-known, this theorem is as follows [37]: "If at the equilibrium position the force function has an isolated maximum, then such an equilibrium position is stable". By the converse of the Lagrange theorem is understood the affirmative answer to the following question: Will the equilibrium position be unstable if the force function is not a maximum at this position?

In such a formulation the problem turns out to be very difficult, and before Chetaev's investigations it was solved only in special simple cases. In particular, Liapunov first investigated (2, § 25) the case where the expansion of the force function U in the neighborhood of the equilibrium position $q_s = 0$ has the form $U = U_m + U_{m+1} + \dots$ (U_m being a form of degree $m > 2$) and the sign of the force function U for $m = 2$ is determined by the terms of the second order.

Liapunov also showed by his direct method (2, § 16, Example 2) that in each case where at the equilibrium position the force function assumes a minimum and this can be determined from the consideration of the totality of terms of the lowest order in the expansion of the increment of this function in terms of the powers of the increments of the coordinates, instability of the equilibrium takes place. This problem was also investigated by other authors (Hadamard, Painlevé).

In order to solve the problem of the converse of the Lagrange theorem, Chetaev first had to develop the direct method of Liapunov. He gave a general theorem of instability based on the ideas of the method of Liapunov functions. This theorem turned out to be very useful for the solution of the concrete mechanical problem described. However, the importance of the general theorem on instability as given by Chetaev turned out to be considerably wider. This theorem can be considered as the most general and universal criterion of instability.

The original formulation of the theorem [11] was given in 1930. A more general formulation and a modification of the theorem were given in Paper [16]. An expanded proof for the general criteria of instability is given in Paper [23].

The theorem on instability is as follows [16,17]:

If the differential equations of perturbed motion are such that (i) for a certain function V, which admits an infinitely small upper bound, there exists a region in which $VV' > 0$, (ii) if in this region ($VV' > 0$) for certain values of the quantities x_s, numerically small as desired, it is possible to single out a region into which a certain function $W > 0$ which vanishes on the boundary, i.e. $W = 0$, assumes for its total derivative with respect to the time W' values which are all of the same sign, then the unperturbed motion is unstable.

If the region $VV' > 0$ considered in the theorem is bounded by the surface $V = 0$ and besides $V' > 0$ holds, then the function V can be taken for W.

As the function W also V' can be taken. Then we obtain the original formulation of the theorem on instability as given in the Paper [11].

These interesting criteria of instability gave rise to a certain amount of debate. At first, it was thought, incorrectly, that the theorem did not hold in the large. It should be pointed out that the original formulations were given by Chetaev in the shortest possible form and were designed for the investigation of those cases of the equations of perturbed motion for which misunderstandings in the interpretation of such concepts as the regions $VV' > 0$, $V > 0$, $W > 0$ and so on were almost excluded.

Later, in his book [37] and in Paper [48], Chetaev explained how, in the general case, the terms used in the formulation of his criterion should be understood.

In particular, in Paper [48], he pointed out that the regions $V > 0$, $V' > 0$ and so on in the neighborhood of the point $x_s = 0$ should be considered on the closed time interval $[t_0, \infty]$.

The formulation of Chetaev's instability theorem which has been most widely adopted is that given in the book [37,48]. If one does not introduce conditions for the existence of the region $V' > 0$ on the closed interval $[t_0, \infty]$, it can be formulated in the following way [37,48]:

The function $W(x_1, \ldots, x_n, t)$ will be called positive-definite in the region $V > 0$ if it can vanish in this region only on the boundary $V = 0$, and if for an arbitrary positive number ϵ, no matter how small, there exists such a positive number $l \neq 0$ that for all x_s satisfying the condition $V > \epsilon$ and all $t > t_0$ the inequality $W > l$ holds.

Theorem. If the differential equations of the perturbed motion are such that a function V can be found, bounded in the region $V > 0$ and existing for all $t > t_0$ and for arbitrarily small absolute values of the variables x_s, whose derivative V', by virtue of these equations, is positive-definite in the region $V > 0$, then the unperturbed motion is unstable.

The converse of this theorem has been proved and, thus, its universality established.

In Paper [11] Chetaev proposed a solution for the converse of the Lagrange theorem using the Kronecker characteristics. The complexity of this solution induced him to look for a more elementary solution. The results obtained by Chetaev in Paper [17] can be reduced to the following:

Let the system be described by differential equations in the Lagrange form

$$\frac{d}{dt}\left(\frac{\partial F}{\partial x_s'}\right) - \frac{\partial F}{\partial x_s} = 0, \qquad \frac{dx_s}{dt} = x_s' \qquad (s = 1, \ldots, k)$$

where

$$F = \frac{1}{2}\sum_{i=1}^{k}(x_i')^2 + \frac{1}{2}\sum_{ij} v_{ij}x_i'x_j' + U, \qquad v_{ij} = v_{ji}$$

U, v_{ij} are holomorphic functions of x_s vanishing at the equilibrium position $x_s = 0$, and the expansion of U begins with terms of the order not lower than two. If the force function U is a form U_m and can assume positive values, then the equilibrium is unstable.

The proof is based on the investigation of the behavior of the function

$$V = \frac{1}{2}H^2 - \frac{1}{2}\alpha^2\left(\sum_{i=1}^{k} x_i\frac{\partial F}{\partial x_i}\right)^4, \qquad H = F - 2U$$

in a neighborhood of the point $x_s = 0$. Under the conditions of the theorem this function satisfies the conditions of the instability theorem. If the force function $U = U_m + \ldots$ has a minimum and this can be determined by the lowest order terms, then the equilibrium $x_s = 0$ is unstable.

In Paper [23] Chetaev gave a new proof for the converse of the Lagrange theorem in the general case when the force function U is analytic and does not possess a maximum at the isolated equilibrium position. An elementary proof was given in this paper only for the case where the function U is a homogeneous function of degree m or $U = U_m + U_{m+1} + \ldots$ and the positive sign of the functions $U = U_m + U_{m+1} + \ldots$ and $mU_m + (m+1)U_{m+1} + \ldots$ is determined by the terms of the lowest order U_m without any necessity to consider terms of higher orders. It should be noted that the first of the cases considered appears now in text books.

Elementary proofs (in the sense of Chetaev's definition) for other more complicated and subtle cases of the converse of the Lagrange theorem were given in Paper [48].

Here, the following particular cases are considered:

(a) The function $U = U_m + U_{m+1} + \ldots + U_{k-1} + U_k + U_{k+1} + \ldots$, where the forms $U_m, \ldots, U_{k-1}$ are constantly negative, the forms U_{k+1}, $U_{k+2}, \ldots$ are constantly positive while the form U_k is of variable sign. For sufficiently small numerical values of the q_s the function $U_m + U_{m+1} + \ldots + U_{k-1} + U_k$ can be made positive. In this case the instability of the point $x_s = 0$ is proved by means of the function

$$V = -H\Sigma p_s q_s \tag{6.1}$$

which satisfies the conditions of Chetaev's instability theorem.

(b) The equilibrium position $q_s = 0$ is unstable if

$$U = -abq_1^2 + (a+b)q_1q_2^2 - q_2^4 \qquad (b > a > 0)$$

The problem is solved by means of the consideration of the function

$$V = -H\left(q_1p_1 + \frac{1}{2}q_2p_2 + q_3p_3 + \ldots + q_kp_k\right)$$

(c) The equilibrium position $q_s = 0$ is unstable if

$$U = -abq_1^2 + (a+b)q_1q_2^2 - q_2^5 \qquad (b > a > 0)$$

The problem is solved by means of the function (6.1).

(d) The following conditions are satisfied:

(i) for arbitrarily small numerical values of the q_s such that $q_1^2 + \ldots + q_n^2 < l$ there exists a certain region C in which $U > 0$;

(ii) there exist certain functions $f_s(q_1, \ldots, q_n)$, which together with their first partial derivatives are continuous in C which vanish for the zero values of the arguments and which are such that all the principal diagonal minors of the functional determinant

$$\left\| \frac{\partial f_s}{\partial q_r} + \frac{\partial f_r}{\partial q_s} \right\| \qquad (s, r = 1, \cdots, n)$$

are bounded from below by positive numbers of the region C, and the function

$$\sum \frac{\partial U}{\partial q_s} f_s$$

is positive-definite in the region C. In such a case the equilibrium position $q_s = 0$ is unstable. The problem is solved by means of consideration of function (6.1).

7. *Investigation of stability in the first approximation for a nonstationary motion.* A large part of Chetaev's works is devoted to investigations in this field [30,35,37,43,58,63]. These works contain, in particular, important estimates for the solutions of the system of linear approximation which have found an extensive practical application.

Among the papers of this section Papers [30,63] should be singled out, in which the theorems of stability and instability are proved in the first approximation for nonstationary systems. As is well-known, Liapunov established the fundamental theorem of stability by the first approximation for regular systems (2, § 12,13).

In Paper [30] Chetaev proves analogous theorems of instability by the first approximation. "If the system of differential equations in the first approximation is regular and if only one among its characteristic numbers is negative, then the unperturbed motion is unstable. If the system in the first approximation is not regular, then, introducing $s = \lambda_1 + \ldots + \lambda_n$, where the λ_i are the characteristic numbers of the normal system of its solutions, we have $s + \mu = = \sigma$ (μ is the characteristic number of the function $1/\Delta$, $\sigma > 0$). Further, if only one of the characteristic numbers λ_i is negative and less than $(-\sigma)$, then the unperturbed motion is unstable".

The proofs of these theorems are based on the properties of the Liapunov characteristic numbers.

Later, in Paper [63], Chetaev gave new proofs for his own and Liapunov's theorems on stability by the first approximation, using the direct method of Liapunov.

In Paper [35] is proved a theorem which appears in many text-books on the theory of stability, namely, on the smallest characteristic number of a nonstationary system, the coefficients $p_{ij}(t)$ of the linear approximation of which approach certain limits c_{ij} as $t \to +\infty$.

If, as t increases indefinitely, the coefficients $p_{ij}(t)$ tend to definite limits c_{ij}, then the lowest characteristic number of the system coincides with the lowest characteristic number of the limit system.

As a consequence of the theorem the following criterion of stability by the first approximation is obtained:

If the elements of the matrix $\|c_{ij}\|$ are such that the real parts of the roots of the characteristic equation $\|c_{ij} - \delta_{ij}\lambda\| = 0$ are negative, then the unperturbed motion $x_s = 0$ is asymptotically stable.

In the same paper a more general case of a system with variable coefficients is also considered, and a method for the construction of Liapunov functions in the form of a quadratic form with variable coefficients is demonstrated.

This paper can be considered as a source of works on the estimation of the velocity of damping of the transition process in terms of the estimates of Liapunov's quadratic functions $V(t, x_1, \ldots, x_n)$. The criterion given in Paper [35] consists of the following:

For $t \geqslant t_0$ let the equation

$$\Delta(\lambda) = \| p_{sr} - \delta_{sr}\lambda \| = 0$$

have roots $\lambda_1, \ldots, \lambda_n$ for which none of the expressions $m_1\lambda_1 + \ldots + m_n\lambda_n$ vanishes for $m_1 + \ldots + m_n = 2$. Then, by a well-known theorem of Liapunov there exists a form $V = \Sigma\, a_{sr}(t)x_s x_r$, which satisfies the equation in the partial derivatives

$$\sum_{s=1}^{n} \frac{\partial V}{\partial x_s}(p_{s1}x_1 + \ldots + p_{sn}x_n) = x_1^2 + \ldots + x_n^2$$

Assume that for all $t > t_0$ the diagonal minors $D_1, \ldots, D_n$ of the discriminant $D = \|da_{sr}/dt + \delta_{sr}\|$ are not smaller than a certain positive number. Then, according to the Sylvester criterion, the derivative dV/dt by virtue of the initial system will be a positive-definite function. Here the boundedness of the derivatives da_{sr}/dt is assumed. On these assumptions, if V is negative-definite, stability takes place. Moreover, if V assumes an infinitely small upper bound, asymptotic stability takes place. If, however, V admits an infinitely small upper bound and can assume negative values, then instability takes place.

In Paper [43] the problem of stability of the solutions of a linear nonstationary system of equations is also considered. The basis of the method, described in this paper for the construction of the Liapunov function $V(x, t)$, is the following concept: if we denote the initial conditions by x_{sk0}, generating for $t = t_0$ the set of the linearly independent solutions $x_{sk}(t)(t > t_0)$, then the quadratic form $V(x, t)$, satisfying the conditions $V[x_{sk}(t), t] = x_{sk0}$ for $s = 1, \ldots, n$; $t > t_0$ will obviously satisfy the condition $dV/dt = 0$. If this form turns out to be positive-definite, then by virtue of the Liapunov theorem the solution $x_s = 0$ will be stable. In Paper [43] Chetaev justifies the possibility of calculating the coefficients $a_{sr}(t)$ of the form $V(x, t)$ by the method of successive approximations, gives the corresponding formulas and discusses the effectiveness of the proposed method of investigation.

8. Effective methods for the construction of the Liapunov functions. In a series of papers on the application of the method of Liapunov functions to the problems of stability Chetaev proved the effectiveness of this method and also justified the possibility of estimating the properties of the transition process in the system. Here, Chetaev emphasized the fact that for a correct selection of the parameters of the system securing the optimal properties for this system, methods based on the calculation of the integral estimates for the separate trajectories corresponding to the chosen initial conditions, prove to be insufficient and may even lead to considerable errors. Paper [47] aimed to show the inconsistency of the integral estimates for the separate perturbed trajectories for the complete characterization of the optimal properties of linear systems, and to show how true estimates can be arrived at by Liapunov method. The paper considered a linear asymptotically stable system described by the equations

$$\frac{dx_s}{dt} = a_{s1} x_1 + \ldots + a_{sn} x_n \tag{8.1}$$

On the basis of estimates for the largest and smallest values of the Liapunov function V as a quadratic form and its total derivative dV/dt by virtue of system (8.1), an estimate is made on the basis of the sphere of radius one from the above, for the transition time of an arbitrary perturbed trajectory of system (8.1), beginning on a sphere of given radius $A > 0$ and crossing into a previously determined a small sphere, of radius $\epsilon > 0$. Since these estimates are determined by the eigenvalues of the matrices of the form V and its derivative dV/dt, and the relations between these eigenvalues are determined by the coefficients a_{ij} of system (8.1), then by the same token a certain guiding rule for the selection of the parameters of system (8.1) is obtained, which guarantee its greatest effectiveness.

It should be noticed that the significance of this paper falls outside the framework of the concrete problem considered in the given paper. In fact, the general considerations on which the method of estimates is based are obviously applicable to more general cases, namely, when a Liapunov function can be constructed and a connection can be effectively observed between the estimates of the properties (positive-definiteness, upper boundedness) of this function and its total derivative (estimation of negative definiteness) and the parameters of the system under consideration. In addition, there are some very fruitful observations on the study of the properties of the system by the simultaneous study of the changes in its properties and in the properties of the corresponding Liapunov function as the parameters vary.

The method of estimating the properties of linear systems by means of quadratic Liapunov functions has been widely adopted, and a series of investigations have resulted in useful and effective estimates for the velocity of damping of the transition process in nonstationary linear and nonlinear systems.

9. Nonlinear systems for which the problem of stability can be solved correctly by sufficiently simple approximate methods are called 'rough' by Chetaev. A system of this kind is considered in the note [64], the results of which are immediately related to Paper [35].

Let the system of differential equations have the form

$$\frac{dx_s}{dt} = (c_{s1} + \varepsilon f_{s1}) x_1 + \dots + (c_{sn} + \varepsilon f_{sn}) x_n \qquad (s = 1, \dots, n) \tag{9.1}$$

where the c_{sr} are constants, f_{sr} bounded real functions of $x_1, \dots, x_n, t$ for $x_1^2 + \dots + x_n^2 \leqslant A$, $t > t_0$.

If the auxiliary system of equations

$$\frac{dx_s}{dt} = c_{s1}x_1 + \dots + c_{sn}x_n \tag{9.2}$$

satisfies the condition that the roots λ_s of the equation $\| c_{sr} - \delta_{sr}\lambda \| = 0$ are such that for arbitrary non-negative integers m_k we have $m_1\lambda_1 + \dots + m_n\lambda_n \neq 0$ when $m_1 + \dots + m_n = 2$, then the virtue of the Liapunov theorem the equation in partial derivatives

$$\sum_{s=1}^{n} \frac{\partial V}{\partial x_s} (c_{s1}x_1 + \dots + c_{sn}x_n) = -(x_1^2 + \dots + x_n^2) = U(x_1, \dots, x_n) \tag{9.3}$$

determines uniquely the quadratic form

$$V = \frac{1}{2} \sum_{s,\, r} a_{sr} x_s x_r$$

For numerically small $\epsilon > 0$ and a small $\mu > 0$ the derivative dV/dt, by virtue of Equation (9.1), will satisfy the condition

$$-\frac{dV}{dt}-\mu\left(x_1^2+\ldots+x_n^2\right)=\sum_{s,r} h_{sr}x_s x_r>0 \quad \text{for} \quad x_1^2+\ldots+x_n^2>0$$

Thus, the asymptotic stability or instability of the unperturbed motion is determined by the constants c_{sr}. The quantities A and ϵ, for which such a correspondence between the systems (9.1) and (9.2) exists, are determined by the inequalities of Sylvester with respect to the form

$$\sum_{s,r} h_{sr}x_s x_r$$

Chetaev points out the possibility of varying the estimates for the numbers ϵ and A which can be changed on account of the variation of the form $U(x_1, \ldots, x_n)$ on the right-hand side of (9.3), and, by the same token, of obtaining for the optimal selection of U the widest estimates.

In the second part of the paper, Chetaev gives concrete estimates for the largest and smallest deviations of the perturbed variables. These estimates have been widely used in practical calculations. In particular, the estimate

$$x_1^2(t)+\ldots+x_n^2(t)\leqslant c\frac{\varkappa_n}{\varkappa_1}e^{(\lambda'+\varepsilon')t} \tag{9.4}$$

is given for the square of the radius of the sphere, into which at the instant t the point in the perturbed motion $x_s(t)$ will enter under the initial condition $x_{10}^2 + \ldots + x_{n0}^2 = c$ for $t > t_0 = 0$. This estimate generalizes to the case of quasi-linear rough systems the estimate for the velocity of damping of the transition process in linear systems obtained earlier by Chetaev in Paper [47]. In the inequality (9.4) the quantities κ_1 and κ_n denote the largest and the smallest eigenvalues of the quadratic form determining the function V, ϵ' is a sufficiently small positive constant and λ' is the largest root of the equation

$$\left\|\frac{1}{4}\sum_{\beta=1}^{n}\left(\frac{\partial b_{r\beta}}{\partial x_s}x_\beta+\frac{\partial b_{\beta r}}{\partial x_s}x_r\right)-\lambda a_{sr}\right\|=0 \quad \left(\frac{dV}{dt}=\frac{1}{2}\sum_{s,r} b_{sr}x_s x_r\right)$$

Here dV/dt denotes the derivative of the Liapunov function calculated by virtue of system (9.1).

10. Concluding this survey of the investigations of Chetaev into stability, it is necessary to emphasize the importance of his monograph "Stability of motion" [37,52].

This small book contains an investigation into the stability of motion of mechanical systems with a finite number of degrees of freedom. These investigations, which were initiated by the classical works of Liapunov, and continued by the scholars of our country, consist of the systematic application of Liapunov's second method. Chetaev achieved important results in this field.

In this monograph, Chetaev does not attempt a complete exposition of all the achievements in the field, but confines himself to those investigations which have the greatest value in application.

Without enlarging the book, Chetaev managed to include in the second edition [52] new theoretical results as well as a series of new problems illustrating the theoretical results.

Chetaev emphasizes the vital fact that in the definition of stability Liapunov used the concept of number and not the concept of an infinitely small quantity.

This fact permits the successful application of Liapunov's methods to the solution of applied problems on stability which arise with the development of technology and physics.

The author draws attention to the method useful in practical application, proposed by Liapunov in the proof of his theorem for finding the dimensions of the region of the initial perturbations provided an arbitrary positive number δ is given, which determines the region of the phase space, inside which the trajectories of the perturbed motion of the system must lie.

In this book a condition for asymptotic stability is proposed which is somewhat more general than the condition corresponding to the Liapunov theorem.

The influence of perturbation forces on the equilibrium is examined. Theorems of Kelvin about the influence of dissipative and gyroscopic forces on stability are strongly proved. The important concepts of secular and temporal stability introduced by Kelvin are explained.

The established possibility of estimating the characteristic numbers by means of averaging the coefficients has great significance for practical calculations.

Chetaev attached great significance to the correct statement of the problem of stability. As a model of the statement he considered the formulation of the stability problem given by Liapunov. Also, Chetaev always paid great attention to the selection of those variables with respect to which the stability is to be investigated. Here it is necessary to point out that ignorance of this fact sometimes leads to the conclusion that stability problems which can be covered by the concepts of Liapunov's stability theory are considered by some investigators as falling completely outside the framework of this theory. For example, the majority of cases which are of interest in applications of the so-called orbital stability can be covered by Liapunov's definition of stability (a classical example is the motion of a particle in the field of central Newtonian forces (2)) provided that the variables are properly selected.

C. Works on Qualitative Methods in Analysis

One of Chetaev's first works on the qualitative theory of differential equations was his proof of the general criterion of stability of motion in the sense of Poisson.

One criterion of stability in the sense of Poisson was indicated by Poincaré. This criterion required the invariance of the volume of a certain set W in the motion along the trajectories of the system. In his Papers [7,8], Chetaev frees himself from the requirement of the invariance of the volume and proves criteria for periodic functions X_s with respect to time.

11. In Chetaev's works analytical methods are developed for the investigation of the behavior of the qualitative picture of the trajectories of dynamical systems and, in particular, methods which originate in the problems of the change of this qualitative picture as the parameters of the system vary continuously.

Here must be mentioned problems on the theory of bifurcation of equilibrium which are closely connected with the problems of stability and instability of equilibrium. A series of papers connected with these problems is devoted to the theory of the Kronecker characteristics [13, 15,18,22,24].

By the term "Kronecker characteristic" in Chetaev's papers is understood a numerical characteristic of a set of $n+1$ functions $F_0(x_1, \ldots, x_n), \ldots, F_n(x_1, \ldots, x_n)$. Let the functions $F_0, \ldots, F_n$ be single-valued, bounded, continuous together with their first order partial derivatives $F_{jk} = \partial F_j/\partial x_k$, and not vanishing simultaneously at any point $(x_1, \ldots, x_n)$ of the space. Any system of equations $F_s = 0$ obtained from any n functions F_s of such a system has only a finite number of roots, which in the space $x_1, \ldots, x_n$ are represented by certain isolated simple points.

Then the Kronecker characteristic $\chi(F_0, F_1, \ldots, F_n)$ can be defined by the equality

$$\chi(F_0, \ldots, F_n) = \sum_{F_k} \operatorname{sign} \Delta_k \qquad (F_k < 0) \tag{11.1}$$

where Δ_k is the minor corresponding to the element F_k in the first column of the determinant

$$D = \begin{vmatrix} F_0 & F_{01} \dots F_{0n} \\ F_1 & F_{11} \dots F_{1n} \\ \dots & \dots \\ F_n & F_{1n} \dots F_{nn} \end{vmatrix}$$

and the summation on the right-hand side of the equality (11.1) is carried out over the roots lying in the region $F_k < 0$ of the system of equations $F_s = 0 (s \neq k)$.

In Paper [13] Chetaev gave a theorem justifying a method proposed by him for the calculation of the characteristics by means of variation of the functions.

This method consists of a continuous variation of the functions of the given system $F_0, \dots, F_n$ to a system of new functions for which the characteristic can be more easily calculated, and in the counting of the losses and gains of units of the characteristic in such a transformation.

Varying continuously the functions $F_0, \dots, F_n$, the characteristic then, and only then, undergoes a change when all the functions vanish at any one of the "transition points" ζ^k.

Assume that we have a single parameter x_0. In the space $x_0, x_1, \dots, x_n$ the system of equations $F_0 = 0, \dots, F_n = 0$ determines the transition points ζ^k.

If to the initial system of functions corresponds the value of the parameter $x_0 = \alpha$ and to the final system $x_0 = \beta$, and if the parameter x_0 varies monotonically, then the difference between the corresponding characteristics is equal to the sum of the characters of the transition points

$$\chi_\alpha (F_0, \dots, F_n) - \chi_\beta (F_0, \dots, F_n) = \sum_k \chi(\zeta^k)$$

This formula permits us to determine the difference between the characteristics of two arbitrary systems of functions. The proof, for example, given by Chetaev of Poincaré's theorem on the parity of characteristics serves as an application of this theorem.

Paper [22] is a systematic survey of numerous modifications and basic applications of this theory.

In the first chapter are given the definitions of the characteristics and the general theorems of Kronecker. The contents of the second chapter consist of Chetaev's theorems on the calculation of the characteristics [13]. The third chapter is devoted to the sources of the characteristic theory, i.e. to the theorems on the separation of roots. In the fourth chapter, the theory of characteristics is applied to problems connected with Poincaré's work "On curves, defined by differential equations". In the fifth and last chapter Chetaev is concerned with integral expressions of the characteristics.

Chetaev indicated the application of the theory of characteristics to the proofs of various theorems of mathematics (part of which are given in the form of problems in each chapter). In this way can be proved, for example, Gauss' theorem on the number of complex roots of a polynomial, Brouwer's theorem on the fixed points of a continuous mapping of a sphere, the algebraic theorems of Sturm and Hurwitz, and the topological theorems of Euler, Poincaré, Hopf and others.

In Papers [15,18,24] the problem of how far the method of Kronecker's characteristics permits us to extend the solution of the problems in the theory of stability is investigated. In Paper [15] the equations of the perturbed motion are considered

$$\frac{dx_s}{dt} = X_s(x_1, \ldots, x_n, t) \qquad (s = 1, \ldots, n) \tag{11.2}$$

It is shown how the definition of stability according to Liapunov can be expressed in terms of the theory of the Kronecker cahracteristics. The unperturbed motion $x_s = 0$ is stable in the sense of Liapunov if for any number $L > 0$ there exists a number $\epsilon > 0$ such that the characteristic χ_t of the system of functions

$$F_0(z_1, \ldots, z_n) = \sum_{i=1}^{n} z_i^2 - L, \qquad F_i(z_1, \ldots, z_n) = z_i - x_i(t) \qquad (i = 1, \ldots, n)$$

where the $x_i(t)$ are the motions described by the system (11.2) satisfy the equality $\chi_t = 1$ for $t \geqslant t_0$ provided that the perturbations x_i^0 for $t = t_0$ satisfy the condition

$$\chi_{t_0}\left(\sum_{i=1}^{n} z_i^2 - \varepsilon, \quad z_1 - x_1^\circ, \ldots, \quad z_n - x_n^\circ\right) = 1$$

In the opposite case the motion is unstable.

Making use of the formula in Paper [13] mentioned above, it is shown how the variation of χ_t can be expressed in terms of the Kronecker characteristic of a certain new system of functions and how the basic theorems of the direct method of Liapunov can be proved by means of the Kronecker characteristics. Thus, in the case of Liapunov's first theorem of stability, the variation of the characteristic χ_t of the system of functions

$$F_0(z) = V(z_1, \ldots, z_n) - c, \qquad F_i(z) = z_i - x_i(t) \qquad (i = 1, \ldots, n)$$

(V is the Liapunov function) as t changes is given according to the above formula by

$$\chi_{t_0} - \chi_t = \sum_k \operatorname{sing} V' \tag{11.3}$$

where the summation is carried out over the points ξ_k for which $F_j = 0$ ($j = 0, 1, \ldots, n$), and t varies from t_0 to the instant under consideration. If $\{x_i^0\} \in [F_0 < 0]$, then $\chi_{t_0} = 1$ and because of $V' \leqslant 0$ we have $\chi_{t_0} - \chi_t = 0$, i.e. $\chi_t = 1$. This proves Liapunov's theorem. Analogously, in terms of the Kronecker characteristics, Chetaev's general theorem of instability can be proved. It must be mentioned that in this paper Chetaev also considers the converse problem of his instability theorem and indicates a process of constructing a sequence of functions V_k in the region $VV_k' > 0$ for which there exist points in sufficiently small neighborhoods $|\kappa_s| < \epsilon_k$ of the unperturbed motion as $\epsilon_k \to 0$.

In Paper [18] Chetaev clarifies the algebraic nature of the Liapunov method in the theory of stability of motion, and shows how the conditions of stability of motion, expressed in terms of the Kronecker characteristics can be connected with the problems of separating the real roots of algebraic equations.

Let the equations of the perturbed motion have the form (11.2), where X_s are holomorphic functions of x_s, the coefficients being continuous functions of time t, and $X_s(0, \ldots, 0, t) = 0$.

Assume that there exists a positive-definite function

$$V(x_1, \ldots, x_n, t) \geqslant W(x_1, \ldots, x_n) > 0$$

The region of stability is assumed to be defined by the inequality

$$W(x) \leqslant c \qquad (c = \text{const}, \ c > 0) \tag{11.4}$$

Assume that for $t = t_0$ the initial values x_{s0} are selected in the region (11.4) and that for $t \geqslant t_0$ the function $V[x_i(x_{s0}, t), t]$ is denoted by $f(t)$. If, further, $\Phi(y, t)$ denotes a function which, by means of the inequality $\Phi(y, t) < 0$, defines a region bounded by the contour $t = t_0$, $t = T$, $y = -c - \epsilon$, $y = -\epsilon$, then, under the condition

$$\chi(\Phi, yf', f) = 0$$

the motion under consideration $x_s(x_{s0}, t)$ will remain in the region (11.4) for $t \in (t_0, T)$, i.e. it will be stable in the large ($W < c$) on the finite interval of the time $t_0 < t < T$. If, however, $\chi(\Phi, yf', f) > 0$, then in the motion during the time interval from t_0 to T the function V

assumes the value c at least once. Also, a sequence of successive derivatives $f^{(\nu)}$ of the function can be considered, which are obtained by virtue of the equations of the perturbed motion (11.2), and the Kronecker characteristics $\chi(\Phi, yf^{(k-1)}, f^{(k)})$. The basic contents of the paper consist of the following: Chetaev shows that, if for a certain function V by virtue of the differential equations (11.2) the sequence $f, f', \ldots, f^{(k)}$ can be constructed, which in analogy with the well-known algebraic methods he calls a sequence of Budan, and if

$$\chi(\Phi, yf^{(k-1)}, f^{(k)}) = 0$$

then, on the basis of investigating this sequence, we can make conclusions about the stability of the unperturbed motion. The argument is based on the connection between the value of the Kronecker characteristic χ and the number of the losses in the change of the sign in the Budan sequence when passing from t_0 to T. In the problem under consideration this permits us to estimate the number of roots of $V - c = 0$ in the interval $[t_0, T]$ and, consequently, in the case of the absence of the roots, to draw conclusions about the stability of the unperturbed motion. The unperturbed motion is stable if the number of changes of the sign in the Budan sequence when passing from t_0 to $t_1 (t_1 < T)$ is for each such value t_1 either a negative number or zero or an even positive number.

At the end of Paper [18] Chetaev points out the possibility of formulating theorems analogous to the Liapunov theorem and corresponding to more general cases of the Budan sequence $f, f', \ldots, f^{(k)}$.

In Paper [24] Chetaev indicates the possibility of generalization of a problem, connected with the problem of the center and considered earlier by Poincaré, Liapunov and Birkhoff.

12. In Paper [58] Chetaev shows that the estimation problems of approximate integration have much in common with the problems of stability of motion. On the basis of this he develops the method of Liapunov functions for its application to the problem of deducing the above estimates. He considers the system of differential equations (11.2), where the X_s are holomorphic functions of the real variables $x_1, \ldots, x_n$ in a certain region D for all values of time t. Assume that by a certain method of approximate integration an approximate solution.

$$x_s = u_s(t) \qquad (s = 1, \ldots, n) \tag{12.1}$$

of Equations (11.2) is obtained which is to be compared with the true solution $x_s = u_s(t) + \xi_s$. To estimate the differences ξ_s Chetaev makes use of the A, λ- estimate introduced by him in the theory of stability. Given the positive constants A, λ, the approximate solution (12.1) has the A, λ-estimate if, for the initial deviations $\xi_{10}, \ldots, \xi_{n0}$ satisfying the inequality $\xi_{10}^2 + \ldots + \xi_{n0}^2 < \lambda$ for every t larger than t_0 by virtue of Equations (11.2), the condition

$$\xi_1^2 + \cdots + \xi_n^2 < A$$

is satisfied.

In order to deduce the A, λ-estimate Chetaev, as in the case of stability problems, considers the system of equations for the perturbed motion

$$\frac{d\xi_s}{dt} = p_{s1}(t)\,\xi_1 + \cdots + p_{sn}(t)\,\xi_n + f_s \qquad (s = 1, \ldots, n)$$

corresponding to the deviations ("perturbations") ξ_s of the approximate solution $u_s(t)$ from the actual solution $x_s(t)$, and the system of equations for the first approximation

$$\frac{d\xi_s}{dt} = p_{s1}(t)\,x_1 + \cdots + p_{sn}(t)\,\xi_n$$

which he uses for the construction of a quadratic Liapunov function $V(t, \xi_1, \ldots, \xi_n)$. Assume that the coefficients p_{sk} are such that there exists a Liapunov function which admits an infinitely small upper bound, is negative-definite and the inequalities

$$\frac{\partial V}{\partial t} + \sum_{s=1}^{n} (p_{s1}\xi_1 + \cdots + p_{sn}\xi_n)\,\frac{\partial V}{\partial \xi_s} \geqslant \xi_1^2 + \cdots + \xi_n^2$$

$$1 - \sum_{s=1}^{n} \left|\frac{\partial V}{\partial \xi_s}\right| > 0$$

hold for all $t \geqslant t_0$ in the region $\xi_1^2 + \ldots + \xi_n^2 \leqslant A$. Denote by l the greatest lower bound of $|V|$ on the sphere $\xi_1^2 + \ldots + \xi_n^2 = A$. If inside the sphere $\xi_1^2 + \ldots + \xi_n^2 \leqslant \lambda$ the inequality $|V| < l$ holds and in the region $\lambda \leqslant \xi_1^2 + \ldots + \xi_n^2 \leqslant A$ the inequalities $|f_s| \leqslant \lambda$ hold, then the approximation $u_s(t)$ has the A, λ-estimate. The proof of this proposition is deduced from the results of [37], referring to the estimates of the region of admissible initial deviations, by means of Liapunov functions, the latter being quadratic forms.

Two examples are also considered. As is always the case with Chetaev's works, besides the fact that these examples illustrate in concrete form the general methods of the author, they also have an independent interest. In the first example is discussed the possibility of replacing the differential equation of the nth order

$$a_0 \frac{d^n x}{dt^n} + a_1 \frac{d^{n-1} x}{dt^{n-1}} + \cdots + a_n x = 0 \tag{12.2}$$

by the approximate equation of the $(n-1)$th order

$$a_1 \frac{d^{n-1}x}{dt^{n-1}} + \cdots + a_n x = 0$$

obtained from (12.2) by dropping the "inertia" term $a_0 d^n x/dt^n$, where a_0 is small in comparison with the remaining coefficients. Chetaev indicates the possibility of obtaining the A, λ-estimate provided that

$$a_1 p^{n-1} + \cdots + a_n = 0$$

possesses roots with negative real parts. By the same token, in the case of the given example, Chetaev actually demonstrated a device for investigating, by the method of Liapunov, problems of the behavior of solutions of a linear equation with a small parameter in the term containing the highest order derivative.

In the second example Chetaev demonstrated the estimation method for the approximate solution of the equation $dx/dt = X(x, t)$, obtained by a selection of the solution $x(t)$ in the form of a linear expansion $a_0\phi_0(x) + a_1\phi_1(x) + \ldots$ in terms of the functions $\phi_0(t)$, $\phi_1(t)$, ... of a certain family given in advance, and made important observations about this method.

In Paper [39] the extension of the d'Alembert method of integrating linear differential equations with constant coefficients to systems of linear equations is described.

D. Applied Problems

Applied problems always occupied a central position in Chetaev's work.

In Paper [10] Chetaev applied the Liapunov theory of stability to the solution of the problem of lateral stability of an airplane. He obtained sufficient conditions for stability. In the monograph [37] he considered the problem of stability of a rectilinear flight of a neutral airplane with respect to longitudinal motions.

A number of investigations have been concerned with the problem of stability of the rotating motion of projectiles. Maievskii was the first who applied approximate analysis and obtained in 1865 the well-known, and in some sense, necessary condition for the stability of the rotating motion of a projectile in flat trajectories.

In Papers [28,38,57] Chetaev succeeded in solving the problem of the sufficient conditions for the stability of the rotating motion of a projectile.

Chetaev [28] considered first the rectilinear flight of a projectile, assuming that the velocity of the motion of its center of gravity and the angular velocity of rotation are both constant. With these assumptions the problem reduces, as was shown by Maievskii, to the case of Lagrange and Poisson of the motion of a heavy rigid body about a fixed point. The solution depends on the location of the roots of the polynomial

$$f(u) = (\alpha - au)(1 - u^2) - (\beta - br_0u)^2 \qquad (u = \cos\theta)$$

where θ is the angle of nutation. Chetaev showed that the angle of nutation θ will have small deviations from its unperturbed value provided that all the roots of the polynomial $f(u)$ are larger than $1 - \delta$, where δ is a small positive number. All the roots of the polynomial $f(u)$ will be larger than $1 - \delta$, if all the roots of the polynomial $F(x) = - f(1-\delta - x)$ are negative. For this it is necessary and sufficient that the Hurwitz conditions be satisfied. Thus, the latter lead to the sufficient conditions of the stability of the angle of nutation of the projectile.

Chetaev shows that for an ideal gun ($\theta_0 = \theta_0' = 0$) these inequalities are satisfied simultaneously and independently of δ if the Maievskii inequality $b^2r_0^2 - 2a > 0$ is satisfied.

For an actual gun the indicated inequalities determine the value of the corresponding deviation.

Further, Chetaev studies the following cases of the rectilinear flight of a projectile: (i) variable angular velocity of rotation and constant velocity of motion of the center of gravity; (ii) variable velocity of motion of the center of gravity and variable angular velocity of rotation of the projectile.

In both cases Chetaev finds sufficient conditions of stability for the nutation angle of the projectile.

Next, he passes to the planar case of curvilinear motion of the center of gravity of the projectile, subject to the action of overturning and drag couples of forces of the air pressure. In this case the stability problem reduces to the finding of sufficient conditions in order that the region of possible changes of the variable u, determined by a certain equation, be inside the interval $(1 - \delta, 1)$. Because of the lack of sufficient experimental data it is difficult to select the most acceptable majorant. Therefore, Chetaev proposed an approximate method for the analysis of the stability conditions. The essence of this method consists in the consideration of small sections of the trajectory as arcs of the corresponding circles of curvature. In this way he succeeded in obtaining the necessary conditions of stability.

In Paper [38] Chetaev investigates by Liapunov's method the stability of the flight of a projectile in a very flat trajectory The differential equations of motion are taken in the form proposed by A.N. Krylov. Chetaev constructs the Liapunov function in the form of a positive-definite bundle of integrals of the equations of motion, and derives from the conditions, when the construction of such a function V is possible, the conditions for the stability of the unperturbed motion. This method of constructing the Liapunov function in the form of a linear bundle of integrals of the perturbed motion, which allowed him to solve rigorously and completely an important concrete problem, was further developed by Chetaev, and permitted him to solve a series of important problems on the stability of mechanical systems.

In this paper Chetaev explains the reasons for the lack of stability in the flight of projectiles observed in practice. This lack of stability is explained by the action, on the projectile, of dissipative forces with complete dissipation which cannot be taken into account. By means of the construction of a Liapunov function, Chetaev proves rigorously that in the case under consideration the stability of the flight, being dependent on the gyroscopic stabilization of a rotating projectile, is destroyed by dissipative forces.

Paper [57] is a continuation of Chetaev's investigations into the stability of the flight of a projectile. In this paper he considered the case of a projectile, having a cavity, filled continuously with an ideal incompressible fluid. Chetaev considered the solution of this problem to be very important, since in a number of cases, starting from this solution, it is possible to make a sufficient provision for stability against unforeseen negative influences of viscosity.

Chetaev gave a rigorous solution of the problem, in a nonlinear formulation, of the stability of the rotating motions of a projectile with a cavity filled continuously with an ideal fluid and being in the state of irrotational motion without.

In Paper [57] the problem of the stability of the flight of a projectile is considered in the following cases:

(a) The cavity has the form of a circular cylinder, the axis of which coincides with the axis of rotation of the inertia ellipsoid of the projectile (without fluid). Using the results of Zhukovskii (with the well-known extension) Chetaev shows that the problem of the stability of the rotational motions of such a projectile coincides with the classical problem of the stability of the usual projectile, provided that the inertia moments are correspondingly selected. Making use of the results of his previous paper [38], he gives an inequality, the fulfilment of which guarantees the stability of the rotational motions of a projectile, having a cavity filled with a fluid, along flat trajectories.

(b) The cavity has the form of a cylinder with a planar diaphragm. In this case the ellipsoid of inertia of the projectile and the augmented mass, representing the liquid filling of the cavity of the projectile, is three-axial. This circumstance makes it difficult to apply to the problem the results known for continuous rigid projectiles, where naturally, it is assumed that the ellipsoid of inertia is an ellipsoid of revolution. In the paper the equations of Lagrange describing the motion are stated, and the stability of the unperturbed motion in the first approximation is investigated. For the reduced moments of inertia A, B, C inequalities are given, the fulfilment of which guarantees that in the first approximation the roots of the characteristic equation are purely imaginary and the unperturbed motion is stable in the first approximation.

(c) A circular cylindrical cavity where the diaphragms form a cross consisting of two mutually orthogonal diametral planes. In this case, also, the problem is reduced by Chetaev to the classical cases studied by him earlier. On the basis of these calculations, sufficient conditions are given for the stability of the flight of a projectile of the type considered.

This paper of Chetaev proved a starting-point for investigations into the stability of the rotational motions of rigid bodies with cavities, completely filled with a liquid or having a free surface, in general, in a state of vortex motion.

The subject of Paper [38] is close to Paper [49], in which Chetaev solves the problem of stability of rotation about the vertical of a rigid body with a fixed point in the case of Lagrange. The stability is considered with respect to the projections p, q, r of the instantaneous angular velocity of the body on the moving axes and the direction cosines γ_1, γ_2, γ_3 of the vertical. In this paper Chetaev demonstrated the effectiveness of the method proposed by him for the construction of the Liapunov functions in the form of linear bundles of first integrals: namely, by using the known first integrals $V_i = c_i (i = 1, 2, 3, 4)$ of this problem, he constructs Liapunov's function in the form of the quadratic form

$$\begin{aligned} V = V_1 + 2\lambda V_2 - (mgz + Cr_0\lambda) V_3 + \frac{C(C-A)}{A} V_4^2 - \\ - 2C(r_0 + \lambda) V_4 = A(\xi^2 + \eta^2) + 2A\lambda(\xi\alpha + \eta\beta) - \\ - (mgz + Cr_0\lambda)(\alpha^2 + \beta^2 + \delta^2) + 2\lambda C\delta\zeta + \frac{C^2}{A}\xi^2 \end{aligned}$$

where λ is an arbitrary constant. On the basis of the Sylvester conditions it is seen that if the inequality

$$C^2 r_0^2 - 4Amgz > 0$$

is satisfied the constant λ can be chosen in such a way that the function

V is positive-definite. Consequently, this inequality is a sufficient condition for the stability of rotation about the vertical.

This paper [49] by Chetaev, although only one-and-a-half pages long, proved of great significance in mechanics. It stimulated the production of a series of papers in which various problems of stability were solved.

Paper [60] is devoted to the investigation of the motion of a heavy gyroscope in a Cardan suspension, the axis of the inner ring being vertical. Formulating the equations of motion of the gyroscope in the form of the Lagrange equations, and indicating their first integrals, Chetaev reduces the problem to the inversion of hyperelliptic integral. It follows from this solution that in the case of a heavy gyroscope in the Cardan suspension the nutational motions play the leading role.

Further, conditions are indicated for the realization of pseudo-regular and regular precessions of the gyroscope, and conditions are derived for stability with respect to the angle of nutation of the rotation of the gyroscope about the vertical.

The scientific works of Chetaev reflect to a considerable degree the development of analytical mechanics during the last thirty-five years.

Chetaev often said that Galileo, Newton, Lagrange and Liapunov determined the basic stages in the history of mechanics.

The name of Liapunov is associated with the creation of the theory of stability of motion. Up to the beginning of the twentieth century the importance of this problem was not realized. The difficulties connected with its statement made it accessible only to a few distinguished scientists. With this problem Lagrange, Liapunov, Thomson, Tait, Routh, Zhukovskii and Poincaré were concerned.

At the present time this theory has great importance in its applications. The methods of Liapunov and Chetaev are applied to the solution of technical problems in the theory of control, guidance of flight vehicles, construction of instruments and underwater navigation.